

# EIGENVALUES OF THE QCD DIRAC OPERATOR AT FINITE TEMPERATURE AND DENSITY

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We investigate the eigenvalue spectrum of the staggered Dirac matrix in two-color QCD at nonzero temperature and at baryon density when the eigenvalues become complex. The quasi-zero modes and their role for chiral symmetry breaking and the deconfinement transition are examined. The bulk of the spectrum and its relation to quantum chaos is considered. Comparison with predictions from random matrix theory is presented.

## 1 Chiral Condensate

The properties of the eigenvalues of the Dirac operator are of great interest for important features of QCD. The Banks-Casher formula<sup>1</sup> relates the Dirac eigenvalue density  $\rho(\lambda)$  at  $\lambda = 0$  to the chiral condensate,  $\Sigma \equiv |\langle \bar{\psi}\psi \rangle| = \lim_{\varepsilon \rightarrow 0} \lim_{V \rightarrow \infty} \pi \rho(\varepsilon)/V$ . The microscopic spectral density,  $\rho_s(z) = \lim_{V \rightarrow \infty} \rho(z/V\Sigma)/V\Sigma$ , should be given by the appropriate result of random matrix theory (RMT),<sup>2</sup> which also generates the Leutwyler-Smilga sum rules.<sup>3</sup>

A formulation of the QCD Dirac operator at chemical potential  $\mu \neq 0$  on the lattice in the staggered scheme is given by<sup>4</sup>

$$M_{x,y}(U, \mu) = \frac{1}{2a} \sum_{\nu=\hat{x},\hat{y},\hat{z}} [U_\nu(x)\eta_\nu(x)\delta_{y,x+\nu} - \text{h.c.}] + \frac{1}{2a} \left[ U_{\hat{t}}(x)\eta_{\hat{t}}(x)e^{\mu}\delta_{y,x+\hat{t}} - U_{\hat{t}}^\dagger(y)\eta_{\hat{t}}(y)e^{-\mu}\delta_{y,x-\hat{t}} \right], \quad (1)$$

with the link variables  $U$  and the staggered phases  $\eta$ . We report on computations with gauge group  $SU(2)$  on a  $6^4$  lattice at  $\beta = 4/g^2 = 1.3$  and with  $N_f = 2$  flavors of staggered fermions of mass  $m = 0.07$ . For this system the fermion determinant is real and lattice simulations of the full theory with chemical potential become feasible exhibiting a phase transition at  $\mu_c \approx m_\pi/2 \approx 0.3$  where the chiral condensate (nearly) vanishes and a diquark condensate develops.<sup>5</sup>

In the left plot of Fig. 1 we compare the densities of the small eigenvalues at  $\mu = 0$  to 0.4 on our  $6^4$  lattice, averaged over at least 160 configurations. Since the eigenvalues move into the complex plane for  $\mu > 0$ , a band of width  $\epsilon = 0.015$  parallel to the imaginary axis is considered to construct  $\rho(y)$ , i.e.

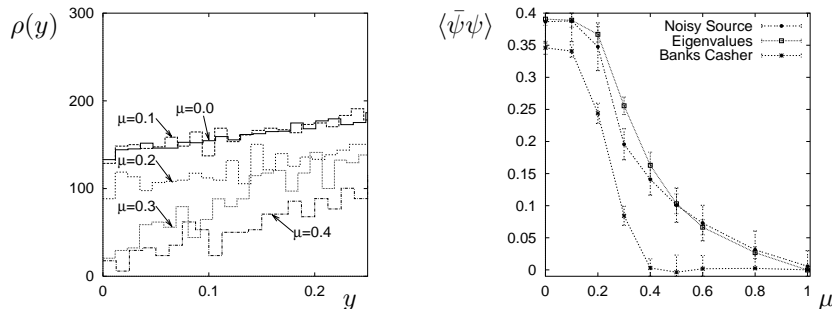


Figure 1: Left plot: Density  $\rho(y)$  of small eigenvalues for two-color QCD on a  $6^4$  lattice from  $\mu = 0$  to 0.4. The loss of quasi-zero modes is accompanied by a vanishing of the chiral condensate. Right plot: Chiral condensate extracted by three different methods (see text).

$\rho(y) \equiv \int_{-\epsilon}^{\epsilon} dx \rho(x, y)$ , where  $\rho(x, y)$  is the density of the complex eigenvalues  $x + iy$ .

The density  $\rho(y)$  is used to estimate a value for the chiral condensate by naively applying the Banks-Casher relation which originally was derived for real eigenvalues appearing in pairs of opposite sign. We further employed the standard definition of the Green's function<sup>3</sup> by inverting the fermionic matrix with a noisy source and by computing its eigenvalues exactly, respectively, to get the condensate. Thus the chiral condensate for two-color QCD with finite chemical potential was extracted by three methods. The preliminary results for  $\langle \bar{\psi}\psi \rangle$  and its variance are shown in the righthand plot of Fig. 1.

## 2 Quantum Chaos

The fluctuation properties of the eigenvalues in the bulk of the spectrum have also attracted attention. It was shown in Ref. 6 for Hermitian Dirac operators that on the scale of the mean level spacing they are described by RMT. For example, the nearest-neighbor spacing distribution  $P(s)$ , i.e. the distribution of spacings  $s$  between adjacent eigenvalues on the unfolded scale, agrees with the Wigner surmise of RMT. According to the Bohigas-Giannoni-Schmit conjecture,<sup>7</sup> quantum systems whose classical counterparts are chaotic have a nearest-neighbor spacing distribution given by RMT whereas systems whose classical counterparts are integrable obey a Poisson distribution,  $P_P(s) = e^{-s}$ . Therefore, the specific form of  $P(s)$  is often taken as a criterion for the presence or absence of “quantum chaos”.

For  $\mu > 0$ , the Dirac operator loses its Hermiticity properties so that its eigenvalues become complex. The aim of the present analysis is to investigate

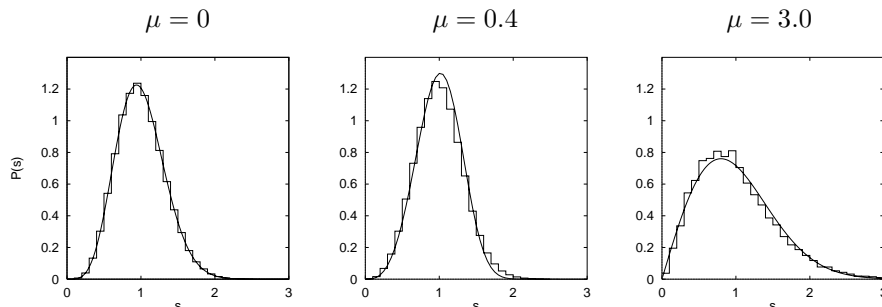


Figure 2: Nearest-neighbor spacing distribution for two-color QCD with varying  $\mu$ . The analytic curves are the Wigner distribution,  $P_W = 262144/(729\pi^3)s^4 \exp(-64/(9\pi)s^2)$  (left), the Ginibre distribution of Eq. (2) (middle) and the Poisson distribution of Eq. (3) (right).

whether non-Hermitian RMT is able to describe the fluctuation properties of the complex eigenvalues of the QCD Dirac operator. We apply a two-dimensional unfolding procedure<sup>8</sup> to separate the average eigenvalue density from the fluctuations and construct the nearest-neighbor spacing distribution,  $P(s)$ , of adjacent eigenvalues in the complex plane. Adjacent eigenvalues are defined to be the pairs for which the Euclidean distance in the complex plane is smallest. The data are then compared to analytical predictions of the Ginibre ensemble<sup>9</sup> of non-Hermitian RMT, which describes the situation where the real and imaginary parts of the strongly correlated eigenvalues have approximately the same average magnitude. In the Ginibre ensemble, the average spectral density is already constant inside a circle and zero outside. In this case, unfolding is not necessary, and  $P(s)$  is given by<sup>10</sup>

$$P_G(s) = c p(cs), \quad p(s) = 2s \lim_{N \rightarrow \infty} \left[ \prod_{n=1}^{N-1} e_n(s^2) e^{-s^2} \right] \sum_{n=1}^{N-1} \frac{s^{2n}}{n! e_n(s^2)}, \quad (2)$$

where  $e_n(x) = \sum_{m=0}^n x^m/m!$  and  $c = \int_0^\infty ds s p(s) = 1.1429\dots$ . For uncorrelated eigenvalues in the complex plane, the Poisson distribution becomes<sup>10</sup>

$$P_P(s) = \frac{\pi}{2} s e^{-\pi s^2/4}. \quad (3)$$

This should not be confused with the Wigner distribution for a Hermitian operator.<sup>6</sup>

Our results for  $P(s)$  are presented in Fig. 2. As a function of  $\mu$ , we expect to find a transition from Wigner to Ginibre behavior in  $P(s)$ . This was clearly seen in color-SU(3) with  $N_f = 3$  flavors and quenched chemical potential,<sup>8</sup>

where differences between both curves are more pronounced. For the symplectic ensemble of color-SU(2) with staggered fermions, the Wigner and Ginibre distributions are very close to each other and thus harder to distinguish. They are reproduced by our preliminary data for  $\mu = 0$  and  $\mu = 0.4$ , respectively. Even in the deconfined phase, where the effect of the chemical potential might order the system, the gauge fields retain a considerable degree of randomness, which apparently gives rise to quantum chaos.

For  $\mu > 1.0$ , the lattice results for  $P(s)$  deviate substantially from the Ginibre distribution and could be interpreted as Poisson behavior, corresponding to uncorrelated eigenvalues. A plausible explanation of the transition to Poisson behavior is provided by the following two (related) observations. First, for large  $\mu$  the terms containing  $e^\mu$  in Eq. (1) dominate the Dirac matrix giving rise to uncorrelated eigenvalues. Second, for large  $\mu$  the fermion density on the finite lattice reaches saturation due to the limited box size and the Pauli exclusion principle.

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